Determination of an unknown diffusion coefficient in a semilinear parabolic problem

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Abstract

A semilinear parabolic problem of second order with an unknown diffusion coefficient in a subregion is considered. The missing data are compensated by a total flux condition through a given surface. The solvability of this problem is proved. A numerical algorithm based on Rothe's method is designed and the convergence of approximations towards the solution is shown. The results of numerical experiments are discussed.

Keywords: parabolic IBVP, parameter identification, nonlocal boundary condition, time discretization, convergence
2010 MSC: 35K20, 35B30

1. Introduction

Recovery of a possible discontinuous diffusion coefficient from boundary measurements of solutions can be found in many applications, such as heat conduction and hydrology. The complete inverse problem is ill posed, so a numerical solution is quite difficult. Spontaneous potential (SP) well-logging is an important technique to detect parameters of the formation in petroleum exploitation. The SP log is a measurement of the natural potential difference or self potential between an electrode in the borehole and a reference electrode at the surface. No artificial currents are applied. This method has been mathematically studied, such as in [1, 2, 3, 4]. The resistivity can depend on temperature and humidity in some geological formations. This makes the problem of the resistivity identification time-dependent. The aim of this paper is to study the recovery of a diffusion coefficient in a subregion from non-local boundary conditions for a transient problem. We assume that the unknown coefficient can change in time, but its shape in space is known. It should be noted that non-local boundary conditions have already been used for identification of some missing parameters at the boundary, cf. [5, 6].

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz continuous boundary Γ. Ω is split into two non-overlapping parts Ω_0 and $\Omega \setminus \overline{\Omega}_0$. We consider a transient diffusion process in Ω. The diffusion coefficient K takes the form $K = k(t,x)\kappa(t,x)$ for a known κ and $k(t,x) = 1$ for $x \in \Omega \setminus \overline{\Omega}_0$ and $k(t,x) = k(t)$ for $x \in \Omega_0$. Γ is split into three non-overlapping parts, namely Γ_N (Neumann part), Γ_D (Dirichlet part) and Γ_0, where besides a Dirichlet boundary condition (BC) also the total flux through this part is prescribed, i.e.,

$$\begin{cases} \int_{\Gamma_0} -K\nabla u \cdot \boldsymbol{\nu} &= h(t) \quad \text{in } (0,T); \\ u &= U(t) \quad \text{on } (0,T)\times\Gamma_0. \end{cases} \tag{1}$$

We assume that $\overline{\Gamma}_D \cap \overline{\Gamma}_0 = \emptyset$, $meas(\Gamma_0) > 0$.

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The goal of this work is to study the following parabolic initial boundary value problem (IBVP) (1)-(2): Find a couple (K, u) such that ($T > 0$ fixed)

$$\begin{aligned} \partial_t u - \nabla \cdot (K \nabla u) &= f(u) && \text{in } Q_T := (0,T) \times \Omega; \\ u &= g^D && \text{in } (0,T) \times \Gamma_D; \\ -K\nabla u \cdot \boldsymbol{\nu} &= g^N && \text{in } (0,T) \times \Gamma_N; \\ u(0) &= u_0 && \text{in } \Omega. \end{aligned} \tag{2}$$

We use the variational framework. Without loss of generality we assume that $g^D = 0$ and $g^N = 0$. This will increase readability of the text. The suitable choice of a test space is

$$V = \{\varphi \in H^1(\Omega);\ \varphi|_{\Gamma_D} = 0,\ \varphi|_{\Gamma_0} = const\},$$

which is clearly a Hilbert space with the norm $\|u\|_V^2 = \|u\|^2 + \|\nabla u\|^2$, where $\|\cdot\|$ represents the norm in $L_2(\Omega)$.

To prove the existence of a weak solution to problem (1)-(2), we apply the Rothe method (cf. [7]). We use an equidistant time-partitioning with a step $\tau = T/n$, for any $n \in \mathbb{N}$, and introduce the notation $t_i = i\tau$ and for any function z

$$z_i = z(t_i), \qquad \delta z_i = \frac{z_i - z_{i-1}}{\tau}.$$

We suggest the following recursive approximation scheme for $i = 1, \dots, n$; $K_i = k_i \kappa_i$, with the unknown $(k_i, u_i) \in \mathbb{R}_+ \times V$ on each time-step

$$\begin{aligned} \delta u_i - \nabla \cdot (K_i \nabla u_i) &= f(u_{i-1}) && \text{in } \Omega; \\ u_i &= 0 && \text{on } \Gamma_D; \\ -K_i \nabla u_i \cdot \boldsymbol{\nu} &= 0 && \text{on } \Gamma_N; \\ \int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu} &= h_i && \\ u_i &= U_i && \text{on } \Gamma_0. \end{aligned} \tag{3}$$

First, we have to show the existence of (K_i, u_i) for any $i = 1, \dots, n$. Then we derive the stability estimates and finally we pass to the limit for $n \to \infty$ to get the existence of a solution to (1)-(2).

The values $C, \varepsilon, C_\varepsilon$ are generic and positive constants independent of the discretization parameter τ. The value ε is small and $C_\varepsilon = C\left(\varepsilon^{-1}\right)$.

2. A single time-step

We present two different ways for solving (3). In the first one we assume that k_i is given and we look for a solution of

$$\begin{aligned} \delta u_i - \nabla \cdot (K_i \nabla u_i) &= f(u_{i-1}) && \text{in } \Omega; \\ u_i &= 0 && \text{on } \Gamma_D; \\ -K_i \nabla u_i \cdot \boldsymbol{\nu} &= 0 && \text{on } \Gamma_N; \\ \int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu} &= h_i. && \end{aligned}$$

We prove that the trace of u_i on Γ_0 continuously depends on k_i. We seek for such k_i for which $u_i|_{\Gamma_0} = U_i$.

In the second method we solve

$$\begin{aligned} \delta u_i - \nabla \cdot (K_i \nabla u_i) &= f(u_{i-1}) && \text{in } \Omega; \\ u_i &= 0 && \text{on } \Gamma_D; \\ -K_i \nabla u_i \cdot \boldsymbol{\nu} &= 0 && \text{on } \Gamma_N; \\ u_i &= U_i && \text{on } \Gamma_0 \end{aligned} \tag{4}$$

for a given k_i. We prove that the total flux $\int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu}$ through Γ_0 continuously depends on k_i. We seek for such k_i that gives $\int_{\Gamma_0} -K_i \nabla u_i \cdot \boldsymbol{\nu} = h_i$.

We adopt the following assumptions on the data

$$0 < C_0 \le k \le C_1; \tag{5}$$

$$0 < D_0 \le \kappa \le D_1; \tag{6}$$

$$U, h, \kappa \in C([0,T]); \tag{7}$$

$$|f(x) - f(y)| \le C|x-y|, \qquad \forall x, y; \tag{8}$$

$$u_0 \in L_2(\Omega). \tag{9}$$

2.1. Auxiliary problem (10)

Consider the following problem

$$\frac{1}{\tau}(u,\varphi) + (K\nabla u, \nabla\varphi) + h\varphi|_{\Gamma_0} = (f,\varphi) \qquad \varphi \in V. \tag{10}$$

For any given $k > 0$ (recall that $K = k\kappa$) this admits a unique weak solution $u_k \in H^1(\Omega)$, which follows from the theory of linear elliptic equations (cf. [8]).

Uniform bound for u_k. We set $\varphi = u_k$ into (10). Applying the Nečas inequality (see [9])

$$\|z\|_\Gamma^2 \le \varepsilon\,\|\nabla z\|^2 + C_\varepsilon\,\|z\|^2, \qquad \forall z \in H^1(\Omega), \qquad 0 < \varepsilon < \varepsilon_0 \tag{11}$$

and using the uniform bounds (5) and (6) one can easily get

$$\left(\frac{1}{\tau} - C_\varepsilon\right)\|u_k\|^2 + (C_0 D_0 - \varepsilon)\,\|\nabla u_k\|^2 \le C(h^2 + \|f\|^2).$$

Fixing a sufficiently small positive ε we see that for $\tau < \tau_0$

$$\|u_k\|^2 + \|\nabla u_k\|^2 \le C(h^2 + \|f\|^2) \qquad \text{for } C_0 \le k \le C_1.$$

u_k ***depends continuously on*** k. Subtract (10) for $k = \beta$ from (10) for $k = \alpha$ and set $\varphi = u_\alpha - u_\beta$ to get

$$\frac{1}{\tau}\left\|u_\alpha - u_\beta\right\|^2 + \big(\alpha\kappa\nabla(u_\alpha - u_\beta), \nabla(u_\alpha - u_\beta)\big) = \big((\beta-\alpha)\kappa\nabla u_\beta, \nabla(u_\alpha - u_\beta)\big).$$

An obvious calculation implies that

$$\left\|u_\alpha - u_\beta\right\|^2 + \left\|\nabla(u_\alpha - u_\beta)\right\|^2 \le C(\alpha-\beta)^2.$$

Using the trace theorem we deduce that for $\mathcal{T}(k) := u_k|_{\Gamma_0}$ we have

$$|\mathcal{T}(\alpha) - \mathcal{T}(\beta)| \le C\left\|u_\alpha - u_\beta\right\|_{L_2(\Gamma)} \le C\sqrt{\left\|\nabla(u_\alpha - u_\beta)\right\|^2 + \left\|u_\alpha - u_\beta\right\|^2} \le C|\alpha - \beta|.$$

2.2. Auxiliary problem (12)

Consider

$$\frac{1}{\tau}(u,\varphi) + (K\nabla u, \nabla\varphi) = (f,\varphi) \qquad \varphi \in \{\psi \in H^1(\Omega);\ \psi|_{\Gamma_0\cup\Gamma_D} = 0\}. \tag{12}$$

For any given $k > 0$ this admits a unique weak solution $u_k \in H^1(\Omega)$ -cf. [8].

Uniform bound for u_k. We set $\varphi = u_k$ into (12). One can readily get

$$\|u_k\|^2 + \|\nabla u_k\|^2 \le C\,\|f\|^2 \qquad \text{for } C_0 \le k \le C_1.$$

u_k ***depends continuously on*** k. Subtract (12) for $k = \beta$ from (12) for $k = \alpha$ and set $\varphi = u_\alpha - u_\beta$ to get

$$\frac{1}{\tau}\left\|u_\alpha - u_\beta\right\|^2 + \left(\alpha\kappa\nabla(u_\alpha - u_\beta), \nabla(u_\alpha - u_\beta)\right) = \left((\beta-\alpha)\kappa\nabla u_\beta, \nabla(u_\alpha - u_\beta)\right),$$

which implies

$$\left\|u_\alpha - u_\beta\right\|^2 + \left\|\nabla(u_\alpha - u_\beta)\right\|^2 \le C(\alpha-\beta)^2.$$

Take any smooth function Φ such that $\Phi|_{\Gamma_D} = 0$ and $\Phi|_{\Gamma_0} = 1$. We recall that $\overline{\Gamma}_D \cap \overline{\Gamma}_0 = \emptyset$. Therefore, the existence of such a function is guaranteed by [10, Lemma 5.1]. Then

$$\Psi(k) := (-k\kappa\nabla u_k \cdot \boldsymbol{\nu}, 1)_{\Gamma_0} = -\frac{1}{\tau}(u_k, \Phi) - (k\kappa\nabla u_k, \nabla\Phi) + (f, \Phi)$$

obeys

$$|\Psi(\alpha) - \Psi(\beta)| = \left|\frac{1}{\tau}\left(u_\alpha - u_\beta, \Phi\right) + \left(\alpha\kappa\nabla(u_\alpha - u_\beta), \nabla\Phi\right) + \left((\alpha-\beta)\kappa\nabla u_\beta, \nabla\Phi\right)\right| \le C|\alpha - \beta|.$$

2.3. Solvability of (3)

A simple consequence of Subsections 2.1 and 2.2 reads as

Lemma 1. *Assume (5)-(9). If* $U(t) \in \mathcal{T}([C_0, C_1])$ $\forall t \in [0,T]$ *or* $h(t) \in \Psi([C_0, C_1])$ $\forall t \in [0,T]$*, then there exist a* $\tau_0 > 0$ *and a couple* $(k_i, u_i) \in \mathbb{R}_+ \times V$ *which solves (3) for* $\tau < \tau_0$.

3. Convergence

The variational formulation of (3) reads as

$$\begin{aligned} (\delta u_i, \varphi) + (K_i\nabla u_i, \nabla\varphi) + h_i\varphi|_{\Gamma_0} &= (f(u_{i-1}), \varphi) \qquad \varphi \in V \\ u_i|_{\Gamma_0} &= U_i. \end{aligned} \tag{13}$$

According to Lemma 1 we see that (13) has a solution on each t_i. The next step is the stability analysis.

Lemma 2. *Let the assumptions of Lemma 1 be fulfilled. Then*

$$\max_{1\le i\le n}\|u_i\|^2 + \sum_{i=1}^{n}\|u_i - u_{i-1}\|^2 + \sum_{i=1}^{n}\|\nabla u_i\|^2\tau \le C.$$

Proof. Set $\varphi = u_i\tau$ into (13) and sum it up for $i = 1, \ldots, j$ keeping $1 \le j \le n$. We obtain

$$\frac{1}{2}\left(\left\|u_j\right\|^2 - \|u_0\|^2 + \sum_{i=1}^{j}\|u_i - u_{i-1}\|^2\right) + \sum_{i=1}^{j}(K_i\nabla u_i, \nabla u_i)\,\tau = \sum_{i=1}^{j}(f(u_{i-1}), u_i)\,\tau - \sum_{i=1}^{j}h_iU_i\tau.$$

Using the Cauchy and Young inequalities we readily get

$$\left\|u_j\right\|^2 + \sum_{i=1}^{j}\|u_i - u_{i-1}\|^2 + \sum_{i=1}^{j}\|\nabla u_i\|^2\tau \le C\left(1 + \sum_{i=1}^{j}\|u_i\|^2\tau + \|u_0\|^2 + \sum_{i=1}^{j}h_i^2\tau + \sum_{i=1}^{j}U_i^2\tau\right).$$

An application of Gronwall's lemma implies that

$$\left\|u_j\right\|^2 + \sum_{i=1}^{j}\|u_i - u_{i-1}\|^2 + \sum_{i=1}^{j}\|\nabla u_i\|^2\tau \le C,$$

which is valid for all $1 \le j \le n$. From this we conclude the proof.

Let us denote by V^* the dual space to V. Then

Lemma 3. *Let the assumptions of Lemma 2 be fulfilled. Then*

$$\sum_{i=1}^{n} \|\delta u_i\|_{V^*}^2 \, \tau \le C.$$

Proof. The relation (13) gives

$$(\delta u_i, \varphi) = (f(u_{i-1}), \varphi) - (K_i \nabla u_i, \nabla \varphi) - h_i \varphi|_{\Gamma_0}.$$

A standard argumentation yields

$$|\,(\delta u_i, \varphi)\,| \le C\,(1 + |h_i| + \|\nabla u_i\|)\,\|\varphi\|_V\,,$$

which implies

$$\|\delta u_i\|_{V^*} = \sup_{\substack{\varphi \in V \\ \|\varphi\|_V \le 1}} |\,(\delta u_i, \varphi)\,| \le C\,(1 + |h_i| + \|\nabla u_i\|)\,.$$

Taking into account Lemma 2 we conclude the proof.

The variational formulation of (1)-(2) reads as: Find (K, u) such that

$$(\partial_t u, \varphi) + (K \nabla u, \nabla \varphi) + h \varphi|_{\Gamma_0} = (f(u), \varphi) \qquad \varphi \in V \tag{14a}$$

$$u|_{\Gamma_0} = U. \tag{14b}$$

Now, let us introduce the following piecewise linear in time function

$$\begin{aligned} u_n(0) &= u_0 \\ u_n(t) &= u_{i-1} + (t - t_{i-1})\delta u_i \qquad \text{for } t \in (t_{i-1}, t_i] \end{aligned}$$

and a step function $\overline{u}_n$

$$\overline{u}_n(0) = u_0, \qquad \overline{u}_n(t) = u_i, \qquad \text{for } t \in (t_{i-1}, t_i].$$

Similarly we define $\overline{K}_n, \overline{h}_n, \overline{U}_n$. The variational formulation (13) can be rewritten as

$$(\partial_t u_n, \varphi) + \left(\overline{K}_n \nabla \overline{u}_n, \nabla \varphi\right) + \overline{h}_n \varphi|_{\Gamma_0} = (f(\overline{u}_n(t - \tau)), \varphi) \qquad \varphi \in V \tag{15a}$$

$$\overline{u}_n|_{\Gamma_0} = \overline{U}_n. \tag{15b}$$

We want to pass to the limit for $\tau \to 0$ in (15) and to arrive at (14).

Theorem 4. *Let the assumptions of Lemma 1 be fulfilled. Then there exists a weak solution to (14).*

Proof. Take any $\xi \in (0, T)$ and integrate (15) on $(0, \xi)$ to get

$$\int_0^\xi (\partial_t u_n, \varphi) + \int_0^\xi \left(\overline{K}_n \nabla \overline{u}_n, \nabla \varphi\right) + \int_0^\xi \overline{h}_n \varphi|_{\Gamma_0} = \int_0^\xi (f(\overline{u}_n(t - \tau)), \varphi) \qquad \varphi \in V. \tag{16}$$

Using the results of Lemma 2 and applying [11, Thm. 2.13.1], we get the existence of a subsequence of $\overline{u}_n$ (denoted by the same symbol again) such that

$$\lim_{n \to \infty} \overline{u}_n \to u \qquad \text{in } L_2(Q_T).$$

Therefore we also get

$$\overline{u}_n \to u \qquad \text{a.e. in } Q_T. \tag{17}$$

Using Lemma 3 we may write for $\xi \in (t_{i-1}, t_i]$ and $\varphi \in V$ that

$$|(\overline{u}_n(\xi) - u_n(\xi), \varphi)| = \left| \int_\xi^{t_i} (\partial_t u_n, \varphi) \right| \le \sqrt{\int_0^T \|\partial_t u_n\|_{V^*}^2} \, \|\varphi\|_V \, \tau^{\frac{1}{2}}.$$

Hence we have for $\tau \to 0$ and $\varphi \in V$ that

$$\begin{array}{ccccc} \int_0^\xi (\partial_t u_n, \varphi) & = (\overline{u}_n(\xi) - u_0, \varphi) & + & (u_n(\xi) - \overline{u}_n(\xi), \varphi) \\ \downarrow & \downarrow & & \downarrow \\ \int_0^\xi (z, \varphi) & = (u(\xi) - u_0, \varphi) & + & 0. \end{array}$$

This is valid for any $\xi \in [0, T]$, thus $z = \partial_t u$ in $L_2((0,T), V^*)$. It holds

$$\left| \int_0^\xi (f(\overline{u}_n(t-\tau)) - f(\overline{u}_n(t)), \varphi) \right| \le C \int_0^\xi \|\partial_t u_n\| \, \|\varphi\| \, \tau = O\left(\tau^{\frac{1}{2}}\right) \|\varphi\| .$$

Applying (17) we get

$$\lim_{\tau \to 0} \int_0^\xi (f(\overline{u}_n(t)), \varphi) = \int_0^\xi (f(u(t)), \varphi) .$$

Lemma 2 and the reflexivity of $L_2((0,T), V)$ give (for a subsequence)

$$\overline{u}_n \rightharpoonup u \qquad \text{in } L_2((0,T), V).$$

This, together with (11), implies

$$\int_0^T \|\overline{u}_n - u\|_\Gamma^2 \le \varepsilon \int_0^T \|\nabla(\overline{u}_n - u)\|^2 + C_\varepsilon \int_0^T \|\overline{u}_n - u\|^2 \le \varepsilon + C_\varepsilon \int_0^T \|\overline{u}_n - u\|^2 .$$

Passing to the limit for $\tau \to 0$ and applying (17) we obtain

$$\lim_{\tau \to 0} \int_0^T \|\overline{u}_n - u\|_\Gamma^2 \le \varepsilon,$$

which is valid for any small $\varepsilon > 0$. Hence

$$\lim_{\tau \to 0} \int_0^T \|\overline{u}_n - u\|_\Gamma^2 = 0 \qquad \text{and} \qquad \overline{u}_n \to u \text{ a.e. in } (0,T) \times \Gamma.$$

Repeating this consideration for Ω_0 instead of Ω we deduce that

$$\lim_{\tau \to 0} \int_0^T \|\overline{u}_n - u\|_{\partial\Omega_0}^2 = 0 \qquad \text{and} \qquad \overline{u}_n \to u \text{ a.e. in } (0,T) \times \partial\Omega_0. \tag{18}$$

Due to the construction we have that $C_0 \le \overline{k}_n \le C_1$. This yields that $\overline{k}_n \rightharpoonup k$ (for a subsequence) in $L_2((0,T))$. Now, applying the Green theorem and taking a sufficiently smooth φ we deduce that

$$\begin{aligned} \int_0^\xi \left(\overline{K}_n \nabla \overline{u}_n, \nabla\varphi\right) &= \int_0^\xi \overline{k}_n \, (\overline{\kappa}_n \nabla \overline{u}_n, \nabla\varphi)_{\Omega_0} + \int_0^\xi (\overline{\kappa}_n \nabla \overline{u}_n, \nabla\varphi)_{\Omega\setminus\Omega_0} \\ &= \int_0^\xi \overline{k}_n \, (\overline{u}_n, \overline{\kappa}_n \nabla\varphi \cdot \nu)_{\partial\Omega_0} - \int_0^\xi \overline{k}_n \, (\overline{u}_n, \nabla \cdot (\overline{\kappa}_n \nabla\varphi))_{\Omega_0} + \int_0^\xi (\overline{\kappa}_n \nabla \overline{u}_n, \nabla\varphi)_{\Omega\setminus\Omega_0} . \end{aligned}$$

Passing to the limit for $\tau \to 0$ we get

$$\begin{aligned} \lim_{\tau \to 0} \int_0^\xi \left(\overline{K}_n \nabla \overline{u}_n, \nabla\varphi\right) &= \int_0^\xi k \, (u, \kappa \nabla\varphi \cdot \nu)_{\partial\Omega_0} - \int_0^\xi k \, (u, \nabla \cdot (\kappa \nabla\varphi))_{\Omega_0} + \int_0^\xi (\kappa \nabla u, \nabla\varphi)_{\Omega\setminus\Omega_0} \\ &= \int_0^\xi k \, (\kappa \nabla u, \nabla\varphi)_{\Omega_0} + \int_0^\xi (\kappa \nabla u, \nabla\varphi)_{\Omega\setminus\Omega_0} \\ &= \int_0^\xi (K \nabla u, \nabla\varphi) . \end{aligned}$$

Applying the density argument we conclude that

$$\lim_{\tau\to 0}\int_0^{\xi}\left(\overline{K}_n\nabla\overline{u}_n,\nabla\varphi\right)=\int_0^{\xi}(K\nabla u,\nabla\varphi)\qquad\forall\varphi\in V.$$

Collecting all considerations above and passing to the limit for $\tau\to 0$ in (16) we arrive at

$$\int_0^{\xi}(\partial_t u,\varphi)+\int_0^{\xi}(K\nabla u,\nabla\varphi)+\int_0^{\xi}h\varphi|_{\Gamma_0}=\int_0^{\xi}(f(u),\varphi)\qquad\varphi\in V.$$

Differentiation with respect to ξ gives (14a). Taking the limit in (15b) and using (18) we get (14b), which concludes the proof.

4. Numerical experiments

The domain we consider is $\Omega=\left(-\frac{1}{2},1\right)\times(-1,1)$, with $\Omega_0=\left(-\frac{1}{2},0\right)\times(-1,1)$ in $\mathbb{R}^2$. Let the time interval be $[0,1]$, i.e., $T=1$. The boundary Γ is split into three non-overlapping parts, namely Γ_D (right), Γ_N (top and bottom) and Γ_0 (left part of Γ).

We use the second solution method described in Section 2 and define the exact diffusion coefficient as follows

$$K(t,x,y)=\tilde{k}(t)\,\mathbb{I}_{\{x<0\}}+\frac{1}{2}.$$

This is equivalent to setting

$$k(t,x,y)=\begin{cases}\tilde{k}(t)+0.5 & \text{if } (t,x,y)\in\Omega_0;\\ 1 & \text{if } (t,x,y)\in\Omega\setminus\overline{\Omega}_0;\end{cases}\qquad \kappa(t,x,y)=\begin{cases}1 & \text{if } (t,x,y)\in\Omega_0;\\ 0.5 & \text{if } (t,x,y)\in\Omega\setminus\overline{\Omega}_0\end{cases}$$

in the previous notation $K=k\kappa$.

Firstly, we prescribe the exact solution (K,u) as follows

$$K(t,x,y)=(1+\sin(10t))\,\mathbb{I}_{\{x<0\}}+\frac{1}{2};\qquad u(t,x,y)=(1+t)\sin\left(\frac{\pi}{2}(1-x)\right).\tag{19}$$

Remark that we choose a trigonometric discontinuous diffusion coefficient with $\tilde{k}(t)=(1+\sin(10t))$. Some simple calculations with use of the exact solution give the exact data for the numerical experiment

$$g^D=g^N=0;\qquad U(t)=\frac{1+t}{\sqrt{2}};\qquad u_0(x)=\sin\left(\frac{\pi}{2}(1-x)\right).\tag{20}$$

We want to approximate the exact solution (19) given the exact data (20). Therefore, we focus on the determination of $\tilde{k}(t)$. For the recovery of $\tilde{k}(t)$ we need the value of $h(t)$ at each time $t\in[0,1]$, which is given by

$$h(t)=\frac{\pi}{\sqrt{2}}\,(1+t)\,(1.5+\sin(10t))\,.$$

We add an uncorrelated noise to this additional condition in order to simulate the errors present in real measurements. The noise is generated randomly with given magnitude $e=0\%, 0.5\%, 1\%$ and 5%.

For the time discretization we choose an equidistant time partitioning with time-step $\tau=0.02$. Applying the backward Euler difference scheme into (4), we are left with a recurrent system of linear elliptic BVPs for $(K_i,u_i)\approx(K(t_i),u(t_i))\,, i=1,2,\ldots,50$ and $\varphi\in\{\psi\in H^1(\Omega);\ \psi|_{\Gamma_0\cup\Gamma_D}=0\}$

$$\frac{1}{\tau}\,(u_i,\varphi)+(K_i\nabla u_i,\nabla\varphi)=(f_i,\varphi)+\frac{1}{\tau}\,(u_{i-1},\varphi)\,;\qquad u_0=u_0;\tag{21}$$

with

$$(f_i,\varphi) = \left(\sin\left(\frac{\pi}{2}(1-x)\right),\varphi\right) + \left((1.5+\sin(10t_i))\left(\frac{\pi}{2}\right)^2(1+t_i)\sin\left(\frac{\pi}{2}(1-x)\right),\varphi\right)_{\Omega_0} + \left(0.5\left(\frac{\pi}{2}\right)^2(1+t_i)\sin\left(\frac{\pi}{2}(1-x)\right),\varphi\right)_{\Omega\setminus\Omega_0}.$$

The unknown $\tilde{k}_i \approx \tilde{k}(t_i), i = 1,\ldots,50$, is determined by the nonlinear conjugate gradient method. On each time-step $t_i, i = 1,\ldots,50$, we minimalize the functional

$$J(\tilde{k}_i) := \left(\int_{\Gamma_0}(\tilde{k}_i + 0.5)\nabla u_i \cdot \boldsymbol{\nu} - h(t_i)\right)^2.$$

Since this functional J is not convex we can only obtain convergence to a local minimum. Therefore, the initial guess has to be sufficiently close to the actual minimizer of the functional. The starting point for this algorithm on the first time-step is set as $\tilde{k}_1^{(0)} = 1$. The starting points on the following time-steps are different in the various examples. We remark that the algorithm stops after maximum 10 iterations with the prescribed error tolerance 10^{-6}.

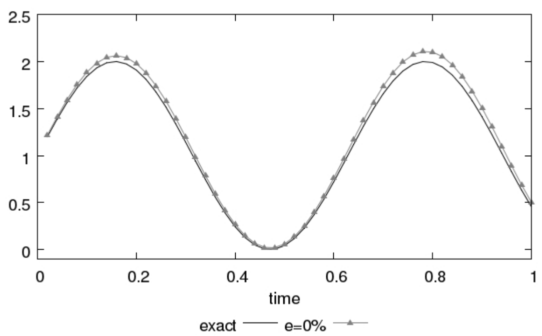
For the space discretization we use a fixed uniform mesh consisting of 144528 triangles. At each time-step, the resulting elliptic BVP (21) is solved numerically by the finite element method (FEM) using first order (P1-FEM) and second order (P2-FEM) Lagrange polynomials.

The results from the recovery of $\tilde{k}(t)$ using the P1-FEM and P2-FEM for the different values of the amplitude e are shown in Fig. 1, 2, 4 and 5. The exact $\tilde{k}(t_i)$ is denoted by a solid line and the approximations $\tilde{k}_i$ by linespoints; $i = 1,\ldots,50$. The evolution of the $\tilde{k}_i$-error for the different time-steps is shown in Fig. 3(a) and 6(a). The $L_2(\Omega)$-error of the approximate solution u_i on $[0,1]$ is depicted in Fig. 3(b) and 6(b).

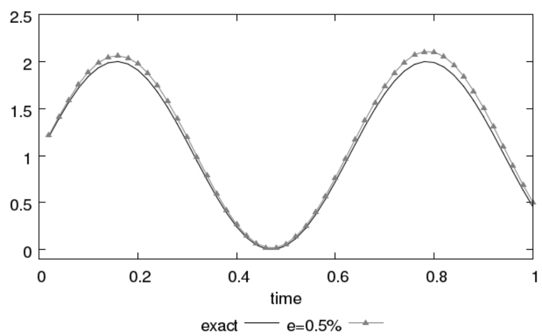
The experiments show that the approximation becomes less accurate with increasing magnitude e when the number of time discretization intervals and the number of triangles in the space discretization is fixed. This result is valid for the P1-FEM as well as for the P2-FEM. We conclude, as expected, that the approximations are more accurate if we use the P2-FEM.

5. Conclusion

A semilinear parabolic problem of second order with an unknown diffusion coefficient in a subregion is considered. The existence of a weak solution for the IBVP is proved when an additional total flux condition through a given surface is prescribed. A numerical algorithm is established and its convergence is demonstrated by a numerical experiment.

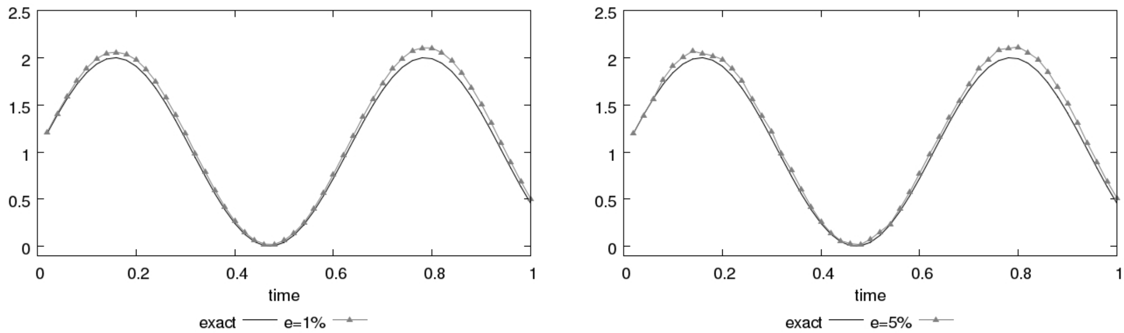


(a)



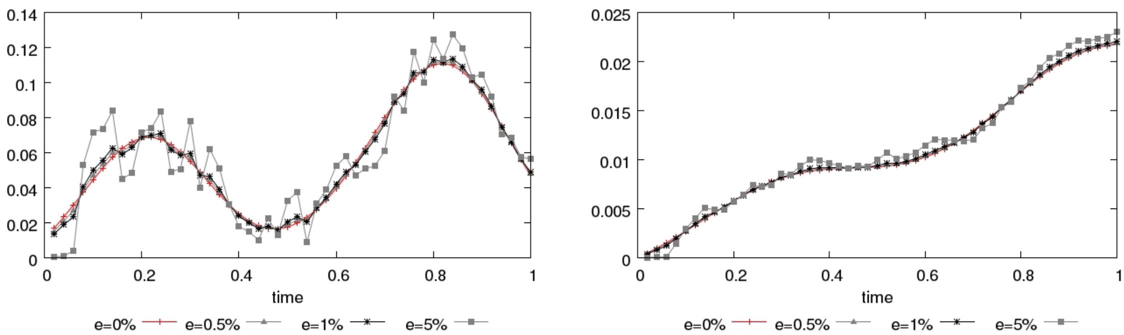
(b)

Figure 1: Numerical value of $\tilde{k}_i$ using the P1-FEM with noise $e = 0\%$ (a) and $e = 0.5\%$ (b); $i = 1,\ldots,50$.



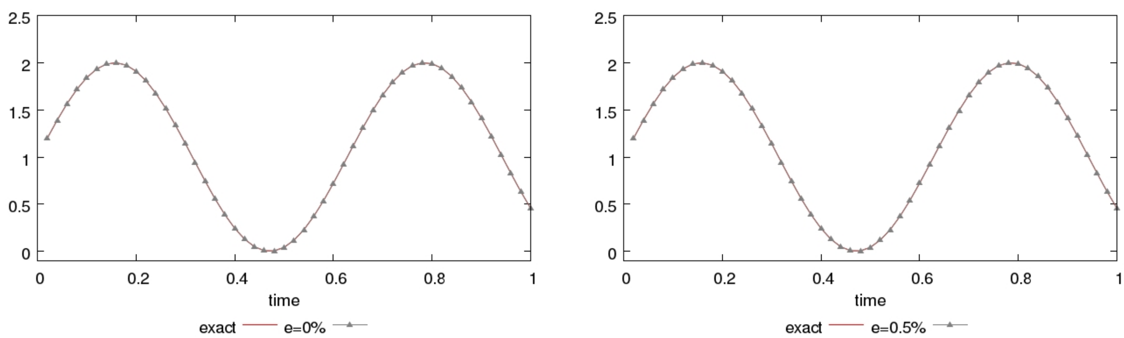
(a) (b)

Figure 2: Numerical value of $\tilde{k}_i$ using the P1-FEM with noise $e = 1\%$ (a) and $e = 5\%$ (b); $i = 1, \ldots, 50$.



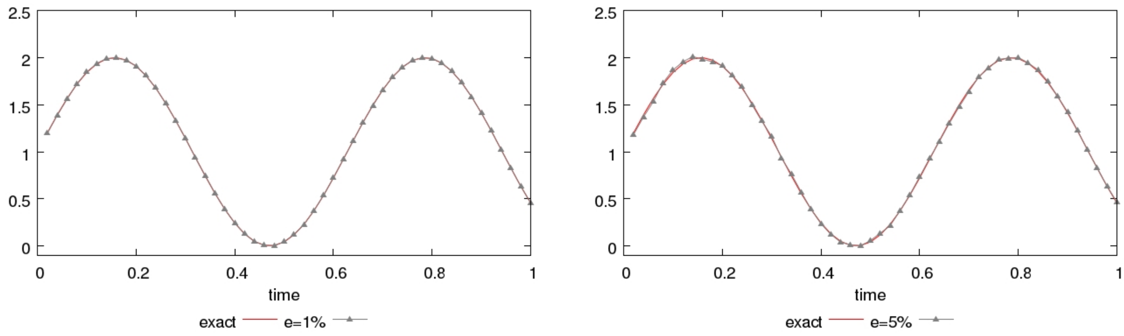
(a) (b)

Figure 3: The absolute $\tilde{k}_i$-error (a) and the absolute $L_2(\Omega)$-error of the approximate solution u_i (b) using the P1-FEM; $i = 1, \ldots, 50$.



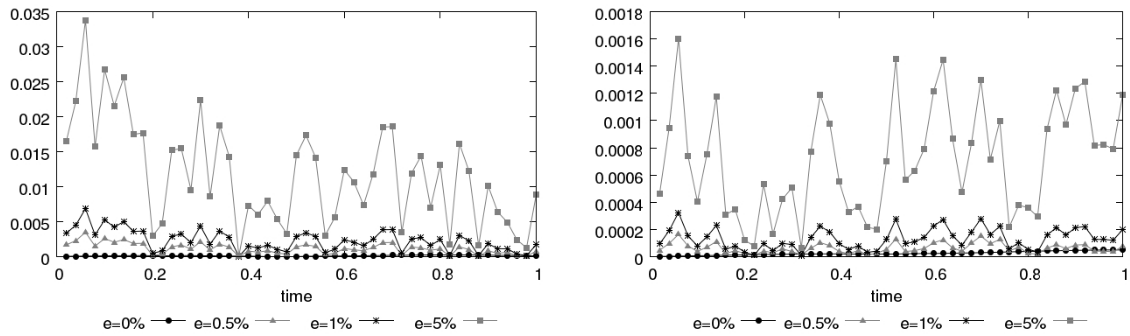
(a) (b)

Figure 4: Numerical value of $\tilde{k}_i$ using the P2-FEM with noise $e = 0\%$ (a) and $e = 0.5\%$ (b); $i = 1, \ldots, 50$.



(a) (b)

Figure 5: Numerical value of $\tilde{k}_i$ using the P2-FEM with noise $e = 1\%$ (a) and $e = 5\%$ (b); $i = 1, \ldots, 50$.



(a) (b)

Figure 6: The absolute $\tilde{k}_i$-error (a) and the absolute $L_2(\Omega)$-error of the approximate solution u_i (b) using the P2-FEM; $i = 1, \ldots, 50$.

Acknowledgement. The research was supported by the BOF/GOA-project no. 01G006B7, Ghent University, Belgium. The authors thank the reviewers for closely reading the manuscript and their useful suggestions.

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